DRAFT FRAMEWORK FOR THE PISA MATHEMATICS DOMAIN

MATHEMATICAL LITERACY

1. The aim of the PISA assessments is to develop indicators of the extent to which the educational systems in participating countries have prepared 15-year-olds to play constructive roles as citizens in society. The assessments are not limited to what students have learned, but focus instead on determining if students can use what they have learned.

DOMAIN DEFINITION

2. The PISA mathematical literacy domain is concerned with the capacities of students to analyse, reason, and communicate ideas effectively as they pose, formulate, solve, and interpret solutions to mathematical problems in a variety of domains and situations. In focusing on real world problems the PISA assessment does not limit itself to the kinds of situations and problems typically encountered in school classrooms. In real world settings, few people are faced with the drill type problems that typically appear in school textbooks and classrooms. Instead, citizens regularly face situations while at work or at leisure, when shopping, travelling, cooking, dealing with their personal finance, judging political issues, and so forth, in which the use of quantitative or spatial reasoning or other mathematical competencies would help clarify, formulate, or solve a problem.

3. PISA mathematical literacy deals with the extent to which 15-year-olds can be regarded as informed and reflective citizens, and intelligent consumers. Citizens in every country are increasingly confronted with a myriad of tasks involving quantitative, spatial, probabilistic, or other mathematical concepts. For example, media outlets (newspapers, magazines, television, and the Internet) are filled with information in the form of tables, charts, and graphs about such subjects as weather, economics, medicine, and sports to name a few. Citizens are currently being bombarded with information on issues such as ‘global warming and the greenhouse effect’, ‘population growth’, ‘oil slicks and the seas’, ‘the disappearing countryside’. Last but not least, citizens are confronted with the need to read forms, pay bills, to successfully carry out transactions involving money, to determine the best buy at the market, and so forth. PISA mathematical literacy focuses on the capacity of 15-year-olds (the age when many students are completing their formal compulsory mathematics learning) to use their mathematical knowledge and understanding to help make sense of these issues and to carry out the relevant tasks.

4. The mathematical literacy definition for PISA is:

Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to engage in mathematics, in ways that meet the needs of that individual's life as a constructive, concerned, and reflective citizen.

5. Some explanatory remarks may help to further clarify this domain definition.

Mathematical literacy

6. The term ‘literacy’ has been chosen to emphasise that mathematical knowledge and skills, as defined within the traditional school mathematics curriculum, do not constitute our primary focus. Instead, the emphasis is on mathematical knowledge put into functional use in a multitude of different situations and contexts in varied, reflective and insight-based ways. Of course, for such use to be possible and viable, a great deal of fundamental mathematical knowledge and skills are needed and such skills form part of our definition of literacy. Literacy in the linguistic sense presupposes but cannot be reduced to, a rich vocabulary and a substantial knowledge of grammatical rules, phonetics, and orthography, and so forth. In the same way, mathematical literacy cannot be reduced to, but certainly presupposes, knowledge of mathematical terminology, facts, and procedures, as well as skills in performing certain operations, and carrying out certain methods.

the world

7. The term ‘the world’ means the natural, social and cultural setting in which the individual lives. As Freudenthal (1983) stated: “Our mathematical concepts, structures, ideas have been invented as tools to organise the phenomena of the physical, social and mental world” (p. ix).

to engage in

8. The term ‘to engage in’ is not meant to cover only physical or social acts in a narrow sense. The term includes also *communicating*, *taking positions towards*, *relating to*, *assessing* and even *appreciating* and *enjoying* mathematics. Thus the definition should not be seen to be limited to the functional use of mathematics in a narrow sense. Preparedness for further study, as well as the aesthetic and recreational elements of mathematics are also encompassed within the definition of mathematical literacy.

life

9. The phrase ‘that individual’s life’ includes his or her private life, occupational life, and social life with peers and relatives, as well as life as a citizen of a community.

10. A crucial capacity that is implied by this notion of mathematical literacy is the ability to pose, formulate, solve, and interpret solutions to problems using mathematics within a variety of situations and contexts. The contexts range from purely mathematical ones to contexts in which no mathematical structure is present or apparent at the outset – the problem poser or solver must successfully introduce the mathematical structure. It is also of importance to emphasise that the definition is not just concerned with knowing mathematics at some minimal level, but it is also about doing and using mathematics in situations that range from the everyday and simple to the unusual and very demanding.

11. Mathematics related attitudes and emotions more generally, such as self-confidence, curiosity, feeling of interest and relevance, and desire to do or understand things, to name but a few, are not components of the definition of mathematical literacy but nevertheless are important contributors to it. In principle it is possible to possess mathematical literacy without possessing such attitudes and emotions at the same time. In practice, however, it is not likely that such literacy is going to be exerted and put into practice by someone who does not have some degree of self-confidence, curiosity, feeling of interest and relevance, and desire to do or understand things that contain mathematical components. The importance of these attitudes and emotions as correlates of mathematical literacy are recognised. They are not components of the mathematical literacy assessment, but will be addressed in other components of PISA.

THEORETICAL BASIS FOR THE PISA MATHEMATICS FRAMEWORK

12. The PISA definition of mathematical literacy is consistent with the broad and integrative theory about the structure and use of language as reflected in recent socio-cultural literacy studies. In James Gee's *Preamble to a Literacy Program* (1998), the term "literacy" refers to the human use of language. One's ability to read, write, listen and speak a language is the most important tool we have through which human social activity is mediated. In fact, each human language and each human use of language has an intricate *design* tied in complex ways to a variety of *functions*. For a person to be literate in a language implies that the person knows many of the design resources of the language and is able to use those resources for several different social functions. Analogously, considering mathematics as a language implies that students not only must learn the design features involved in mathematical discourse (the terms, facts, signs and symbols, procedures, and skills in performing certain operations in specific mathematical sub-domains and the structure of those ideas in each sub-domain), they also must learn to use such ideas to solve non-routine problems in a variety of situations defined in terms of social functions. Note that the design features for mathematics are more than knowing the basic terms, procedures, and concepts that one is commonly taught in schools. It involves how these features are structured and used.

Unfortunately, one can know a good deal about the design features of mathematics without knowing either their structure or how to use those features to solve problems.

13. These scholarly notions involving the interplay of "design features" and "functions" that support the mathematics framework for PISA can be illustrated via the following example.

> The Town Council has decided to construct a streetlight in a small triangular park so that it illuminates the whole park. Where should it be placed?

14. This social problem can be solved following the general strategy used by mathematicians, which the mathematics framework will refer to as *mathematising*. Mathematising can be characterised as having five aspects:

[side bars illustrating each aspect and drawings]

(1) Starting with a problem situated in reality;

Locating where a street light is to be placed in a park.

(2) Organising it according to mathematical concepts;

The park can be represented as a triangle, and illumination from a light as a circle with the street light at its centre.

(3) Gradually trimming away the reality through processes such as making assumptions about what are the important features of the problem, generalising and formalising (which promote the mathematical features of the situation and transform the real problem into a mathematical problem that faithfully represents the situation);

The problem is transformed to locating the centre of a circle that circumscribes the triangle.

(4) Solving the mathematical problem;

Using the fact that the centre of a circle that circumscribes a triangle lies at the point of intersection of the perpendicular bisectors of the triangle's sides, construct the perpendicular bisectors of two sides of the triangle. The point of intersection of the bisectors is the centre of the circle.

(5) And, making sense of the mathematical solution in terms of the real situation.

Relating this finding to the real park. Reflecting on this solution and recognising that if one of the three corners of the park was an obtuse angle, this solution would not be reasonable since the location of the light would be outside the park.

15. It is these processes that characterise how, in a broad sense, mathematicians often *do mathematics*, how people use mathematics in a variety of current and potential occupations, and how informed and reflective citizens should use mathematics to fully and competently engage with the real world. In fact, learning to *mathematise* should be the primary educational goal for all students.

16. Today and in the foreseeable future every country needs mathematically literate citizens to deal with a very complex and rapidly changing society. Accessible information has been growing exponentially, and citizens need to be capable of making decisions about how to deal with this information. Social debates increasingly involve quantitative information to support claims. One example of the need for a citizen to have mathematical literacy is the frequent demand for individuals to make judgements and assess the accuracy of conclusions and claims about information from surveys and studies. Being able to judge the soundness of the claims from such arguments is, and increasingly will be, a critical aspect of being a responsible citizen. Unfortunately, the consequences of the failure to use mathematical notions include confused personal decisions, an increased susceptibility to pseudo-sciences, and poorly informed decision-making in professional and public life.

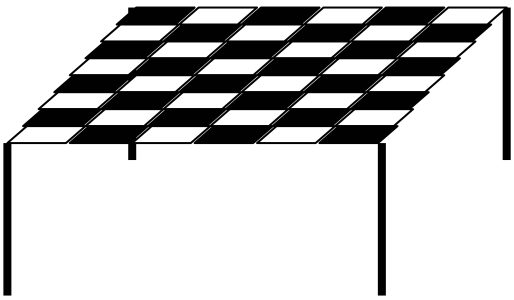
17. A mathematically literate citizen realises how quickly change is taking place and the consequent need to be open to lifelong learning. Adapting to these changes in a creative, flexible and practical way is a necessary condition for a successful citizenship. The skills learned at school will certainly not be sufficient to serve the needs of citizens for the majority of their adult life.

18. The requirements for competent and reflective citizenship are also projected on the needs of the workforce. Workers are less and less expected to carry out repetitive physical chores for all of their working lives. Instead, they are engaged actively in monitoring output from a variety of high-technology machines, dealing with a flood of information, and engaging in team problem solving. The trend is that more and more occupations will require the ability to understand, communicate, use, and explain concepts and procedures based on mathematical thinking. In fact, during the past decade the rate of growth of mathematically based occupations throughout the world is about twice that for all other occupations.

19. Finally, mathematically literate citizens also develop an appreciation for mathematics as a dynamic, changing and relevant discipline that may often serve their needs.

20. The operational problem faced by PISA is how to assess whether 15-year-old students are mathematically literate in terms of their ability to *mathematise.* Unfortunately, in a timed assessment this is difficult because for most complex real situations the full process of proceeding from reality to mathematics and back often involves collaboration, involves locating appropriate resources, and takes considerable time.

21. To illustrate *mathematisation* in an extended problem-solving exercise, consider the 'Fairground' example carried out by an eighth grade class of students (Romberg, 1994):



At a fair, players throw coins onto a board chequered with squares. If a coin touches a boundary, it is lost. If it rolls off the board, it is returned. But if the coin lies wholly within a square, the player wins the coin back plus a prize. What is the probability of winning at this game?

Figure 1 A fairground gameboard

22. Clearly this exercise is situated in reality. First, the students began by realising that the probability of winning depends on the relative sizes of the squares and the coin (identifying the important variables). Next, to transform the real problem to a mathematical problem they realised that it might be better to examine the relationship for a single square and a smaller circle (trimming the reality). Then they decided to construct a specific example (using a problem solving heuristic – "if you cannot solve the problem given, solve one you can"). Note that all the following work was done with respect to this specific example, not the board, the prize, etc. In the example they let the radius of the coin be 3 cm and the side of the squares be 10 cm. They realised that to win, the centre of the coin must be at least 3 cm from each side, otherwise the edge of the coin will fall across the square. The sample space was the square with side 10 cm, and the winning event space was a square with side 4 cm. The relationships are shown in the following diagram.

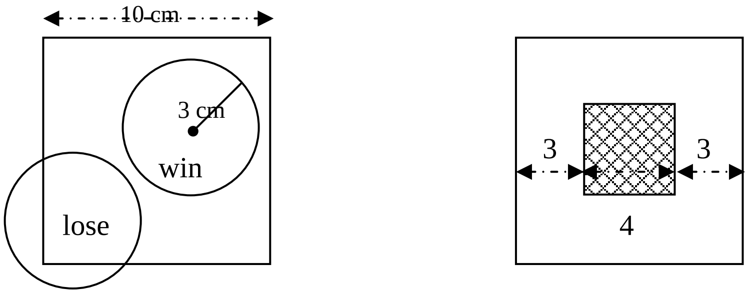


Figure 2 A winning toss and a losing toss (on the left) and the sample and event spaces (on the right)

23. The probability of winning was obtained from the ratio of the area of the sample and event space squares (for the example p =16/100). Then the students examined coins of other sizes, and generalised the problem by expressing its solution in algebraic terms. Finally the students extended this finding to work out the relative sizes of the coin and squares for a variety of practical situations, constructed boards and empirically tested results.

24. Note that each of the five aspects of *mathematisation* is apparent in this solution. Although the problem is complex, all 15-year-old students should understand the mathematical features needed to solve the problem. However, note that in this class the students worked together on this exercise for three days.

25. Ideally, to judge whether 15-year-old students can make use of their accumulated mathematical knowledge to solve mathematical problems confronted as they interact with their world, one would collect information about their capability of *mathematising* a number of such complex situations. Clearly this is impractical. Instead, PISA has chosen to prepare items to assess different parts of this process. The following section describes the strategy chosen to create a set of test items in a balanced manner so that a selected sample of items will cover the five aspects of *mathematising*, and to use the responses to those items to create a proficiency scale as an indicator of the PISA construct of mathematical literacy.

ORGANISATION OF THE MATHEMATICS FRAMEWORK

26. The PISA Mathematics Framework provides the rationale for, and the description of, an assessment of the extent to which 15-year-olds can handle mathematics in a well-founded manner when confronted with real world problems, or in more general terms: an assessment of how mathematically literate 15-year-olds are. To describe the domain that is assessed more clearly, three components have to be distinguished:

- the *situations and contexts* in which the problems students are to deal with are located,

- the mathematical content that has to be used when solving the problems, organised by means of certain *overarching ideas*, and, most importantly,

- the *competencies* that have to be activated in the process of connecting the real world (in which the problems are generated) with mathematics, and thus solving the problems.

27. These components are represented in visual form in Figure 3 before an explanation of each is provided.

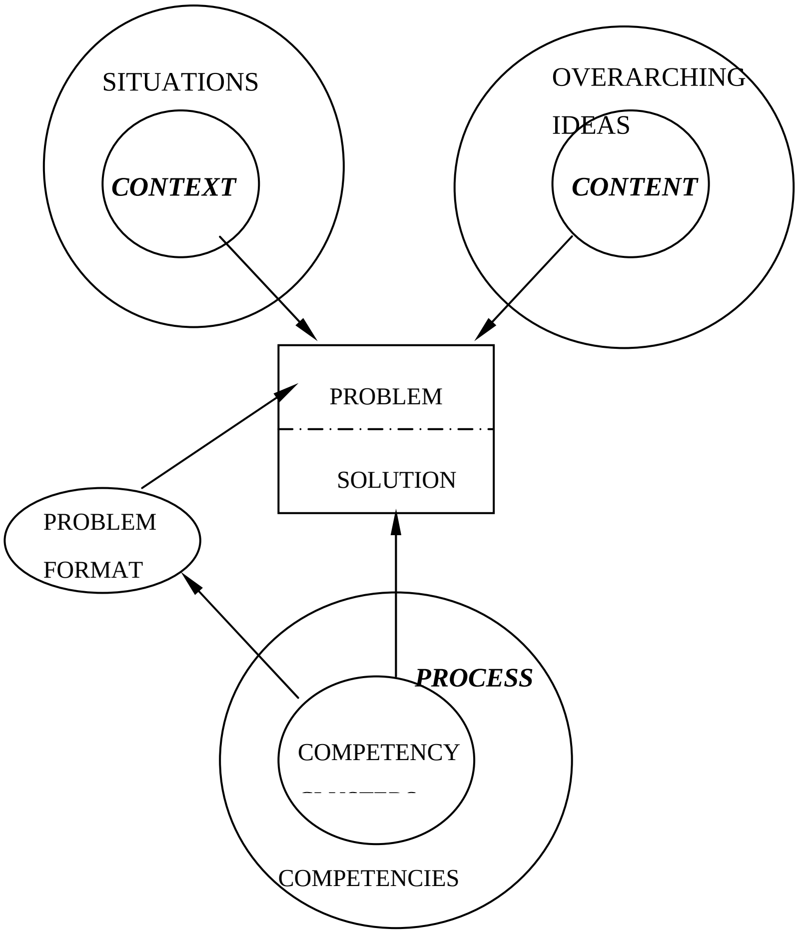


Figure 3 Visualisation of the key components of the Mathematics framework

28. The extent of a person's mathematical literacy is seen in the way in which they use their mathematical knowledge and skills in solving problems for which such mathematical knowledge may be useful. Problems may occur in a variety of *situations and contexts* within the experience of an individual. The problems of interest in PISA are of a broader range than the mathematical problems and applications typically encountered in mathematics as it is taught in schools. PISA problems draw from the real world in two ways. First, problems exist within some broad situations that are relevant to the student's life. The situations form part of the real world and are indicated with a big circle in the upper left of the

picture. Next, within that situation, problems have a more specific context. This is represented by the small circle within the *situations* circle.

[bolded text following as sidebars]

In the above examples the situation is the local community, and the contexts are lighting in a park, and a fairground checkerboard game.

29. The next component of the real world that has to be considered when thinking about mathematical literacy is the mathematical content that a person might bring to bear in solving a problem. The *mathematical content* can be conceived by describing four categories that encompass the kinds of problems that arise through interaction with day-to-day phenomena and that are based on a conception of the ways in which mathematical content presents itself to people. For PISA assessment purposes, these are called 'overarching ideas', namely Quantity, Space and Shape, Change and Relationships, and Uncertainty. This is somewhat different from an approach to content that would be familiar from the perspective of mathematics instruction and the curricular strands typically taught in schools. However, the overarching ideas broadly cover the range of mathematical topics that students are expected to have learned. The overarching ideas are represented by the big circle in the upper right of the diagram in Figure 3. From the overarching ideas the content used in solving a problem is extracted. This is represented by the smaller circle within the overarching ideas circle.

30. The arrows going from the 'context' and 'content' to the problem show how the real world (including mathematics) makes up a problem.

The park problem involves geometrical knowledge related to the ideas of space and shape, and the fairground problem involves (at least in its initial stages) dealing with uncertainty and applying knowledge of probability.

31. The mathematical processes that students apply as they attempt to solve problems are referred to as *mathematical competencies*. These have been described in three *competency clusters* according to different cognitive processes that are needed to solve different kinds of problems. These clusters are a reflection of the way that mathematical processes are typically employed when solving problems that arise as students interact with their world.

32. Thus the Process component of this framework is represented in Figure 3 first by the large circle, representing the general mathematical competencies, and a smaller circle that represents three competency clusters, that we will elaborate below. The particular competencies that are needed to solve a problem will be related to the nature of the problem, and the competencies used will be reflected in the solution found. This interaction is represented by the arrow from the competency clusters to the problem.

33. The remaining arrow goes from the competency clusters to the problem format. The competencies employed in solving a problem is related to the form of the problem.

34. It should be emphasised that the three different components just described are of a different nature. While situations and contexts define the real world problem areas, and overarching ideas reflect the way in which we look at the world with "mathematical glasses", the competencies are the core of mathematical literacy. Only when certain competencies are available to students will they be in a position to successfully solve given problems. Assessing mathematical literacy includes assessing to what extent students possess mathematical competencies they can productively apply in problem situations.

35. In the following sections, these three components are described in more detail.

SITUATIONS AND CONTEXTS

36. An important aspect of the definition of mathematical literacy is using and doing mathematics in a variety of situations. It has been recognised that in dealing with issues that lend themselves to a mathematical treatment, the choice of mathematical methods and representations is often dependent on the situations in which the problems are presented.

37. The situation is the part of the student's world in which the tasks are placed. It is located at a certain *distance* from the students. For PISA, the closest situation is the student's personal life, next is school life, then work and leisure, followed by the local community and society as encountered in daily life. Furthest away are scientific situations. Five situation-types will be defined and used for problems to be solved: personal, educational, occupational, public, and scientific.

38. The context of an item is its specific setting within a situation. It includes all the detailed elements used to formulate the problem.

39. Consider the following example:

> 1000 zed is put into a savings account at a bank. There are two choices : one can get a rate of 4% OR one can get an immediate 10 zed bonus from the bank, and a 3% rate. Which option is better after one year ? After two years ?

40. The situation of this item is 'finance and banking', which is a situation from the local community and society that PISA would classify as 'public'. The context of this item concerns money (zeds) and interest rates for a bank account.

41. Note that this kind of problem is one that could be part of the actual experience or practice of the participant in some real-world setting. It provides an *authentic* context for

the use of mathematics, since the application of mathematics in this context would be genuinely directed to solving the problem. This can be contrasted with problems frequently seen in school mathematics texts where the main purpose is to practise the mathematics involved, rather than to use mathematics to solve a real problem. This *authenticity* in the use of mathematics is an important aspect of the design and analysis of items for PISA, strongly related to the definition of mathematical literacy.

42. It should also be noted that there are some made-up elements of the problem – the money involved is fictitious. This fictitious element is introduced to ensure that students from certain countries are not given an unfair advantage.

43. The situation and context of a problem can also be considered in terms of the distance between the problem and the mathematics involved. If a task refers only to mathematical objects, symbols or structures, and makes no reference to matters outside the mathematical world, the context of the task is considered as intra-mathematical. A limited range of such tasks will be included in PISA, where the close link between the problem and the underlying mathematics is made explicit in the problem context. More typically, problems encountered in the day-to-day experience of the student are not stated in explicit mathematical terms. They refer to real world objects. These task contexts are called 'extra-mathematical', and the student must translate these problem contexts into a mathematical form. Generally speaking, PISA puts an emphasis on tasks that might be encountered in some real-world situation and possess an authentic context that influences the solution and its interpretation. Note that this does not preclude the inclusion of tasks in which the context is hypothetical, as long as the context has some real elements, is not too far removed from a real-world situation, and for which the use of mathematics to solve the problem would be authentic, such as the following example:

> Would it be possible to establish a coinage system based on only the denominations 3 and 5? More specifically, what amounts could be reached on that basis? Would such a system be desirable?

Figure 4 A problem with a hypothetical context that is 'extra-mathematical'

44. This problem derives its quality not in the first place from its closeness to the real world, but from the fact that it is mathematically interesting and calls on competencies that are related to mathematical literacy. The use of mathematics to explain hypothetical scenarios and explore potential systems or situations, even if these are unlikely to be carried out in reality, is one of its most powerful features.

45. In summary PISA values most highly tasks that could be encountered in one of a variety of real-world situations, and that have a context for which the use of mathematics to solve the problem would be authentic. Problems with extra-mathematical contexts that influence the solution and its interpretation are preferred as a vehicle for assessing mathematical literacy, since these problems are most like those encountered in day-to-day life.

MATHEMATICAL CONTENT – THE FOUR 'OVERARCHING IDEAS'

INTRODUCTION

46. Mathematical concepts, structures and ideas have been invented as tools to organise the phenomena of the natural, social and mental world. In schools, the mathematics curriculum has been logically organised around content strands (e.g. arithmetic, algebra, geometry) that reflect historically well-established branches of mathematical thinking. However, in the real world the phenomena that lend themselves to mathematical treatment do not come so logically organised. Rarely do problems arise in ways and contexts that allow their understanding and solution to be achieved through an application of knowledge from a single content strand. The 'fairground' problem described earlier provides an example of a problem that draws on quite diverse mathematical areas.

47. Since the goal of PISA is to assess students' capacity to solve real problems, our strategy has been to define the range of content that will be assessed using a phenomenological approach to describing the mathematical concepts, structures, or ideas. This means describing content in relation to the phenomena and the kinds of problems for which it was created. This approach ensures a focus in the assessment that is consistent with the domain definition, yet covers a range of content that includes what is typically found in other mathematics assessments and in national mathematics curricula.

48. A phenomenological organisation for mathematical content is not new. Two well known publications: *On the Shoulders of Giants: New Approaches to Numeracy* (Steen, 1990) and *Mathematics: The Science of Patterns* (Devlin, 1994) have described mathematics in this manner. However, a variety of ways of labeling the approach and naming the different phenomenological categories can be observed. Suggestions for labeling the approach have included: 'deep ideas', 'big ideas', or 'fundamental ideas'; 'overarching concepts', 'underlying concepts' or 'major domains'; or 'Problematique.' In the mathematics framework for PISA 2003 the label 'overarching ideas' will be used.

49. A variety of possible mathematical overarching ideas can be identified. To quote the above mentioned publications alone leads to: Pattern, Dimension, Quantity, Uncertainty,

Shape, Change, Counting, Reasoning and Communication, Motion and Change, Symmetry and Regularity, and Position. Which should be used for the PISA Mathematics Framework? For the purpose of focusing the PISA mathematical literacy domain it is important that a selection of problem areas is made which grows out of historical developments in the mathematics domain, which encompasses sufficient variety and depth to reveal the essentials of mathematics, and at the same time represents or includes the conventional mathematical curricular strands in an acceptable way.

50. For centuries mathematics was predominantly the science of numbers, together with relatively concrete geometry. The period up to 500 B.C. in Mesopotamia, Egypt and China was the time when the concept of number was established, and operations with numbers and quantities, including quantities resulting from geometrical measurements, were developed. From 500 B.C. to 300 A.D. was the era of Greek mathematics which focussed primarily on the study of Geometry as an axiomatic theory. The Greeks were responsible for redefining mathematics as a unified science of number and shape. The next major change took place between 500 and 1300 A.D. in the Islamic world, India, and China, which established algebra as a branch of mathematics. This founded the study of relationships. With the independent inventions of calculus (the study of change, growth, and limit) by Newton and Leibniz in the 17th Century, mathematics became an integrated study of number, shape, change and relationships.

51. The nineteenth and twentieth centuries saw explosions of mathematical knowledge and of the range of phenomena and problems that could be approached by means of mathematics These include aspects of randomness, and indeterminacy. These developments made it increasingly difficult to give simple answers to the question 'what is mathematics?' At the time of the new millenium many see mathematics as the science of patterns (in a general sense). So a choice of overarching ideas can be made that reflects these developments: patterns in quantity, patterns in shape and space, patterns in change and relationships form central and essential concepts for any description of mathematics, and they form the heart of any curriculum, whether at high school, college or university. But to be literate in mathematics means more. Dealing with uncertainty from a mathematical and scientific perspective is essential. For this reason, elements of probability theory and statistics give rise to the fourth overarching idea, uncertainty.

52. The following list of overarching ideas, therefore, is used in PISA 2003 to meet the requirements of historical development, coverage of the domain, and reflection of the major threads of school curriculum:

- Quantity

- Space and Shape
- Change and Relationships
- Uncertainty.

53. With these four, mathematical content is organised into a sufficient number of areas to help ensure a spread of items across the curriculum, but at the same time a small enough number to avoid a too fine division that would work against a focus on problems based in real situations.

54. The basic conception of an overarching idea is an encompassing set of phenomena and concepts that make sense and can be encountered within and across a multitude of quite different situations. By their very nature each overarching idea can be perceived as a sort of general notion dealing with some generalised content dimension. This implies that the overarching ideas cannot be sharply delineated vis-à vis one another[1]. Rather, each of them represents a certain perspective, or point of view, which can be thought of as possessing a core, a centre of gravity, and somewhat blurred outskirts that allow for intersection with other overarching ideas. In principle, any overarching idea intersects any other overarching idea. The four overarching ideas are summarised in the following section and discussed more fully in 錯誤! 找不到參照來源。.

QUANTITY

55. This overarching idea focuses on the need for quantification in order to organise the world. Important aspects include an understanding of relative size, to recognise numerical patterns, and to use numbers to represent quantities and quantifiable attributes of real world objects (measures). Furthermore, Quantity deals with the processing and understanding of numbers that are represented to us in various ways.

56. An important aspect of dealing with Quantity is quantitative reasoning. Essential components of Quantitative reasoning are: number sense, representing numbers in various ways, understanding the meaning of operations, having a feel for the magnitude of numbers, mathematically elegant computations, mental arithmetic, and estimating.

SPACE AND SHAPE

57. Patterns are encountered everywhere around us: spoken words, music, video, traffic, building constructions and art. Shapes can be regarded as patterns: houses, office buildings,

[1] And, of course, nor can traditional mathematics content strands.

bridges, starfish, snowflakes, town plans, cloverleaves, crystals and shadows. Geometric patterns can serve as relatively simple models of many kinds of phenomena, and their study is possible and desirable at all levels (Grünbaum, 1985).

58. In the study of shape and constructions, we are looking for similarities and differences as we analyse the components of form and recognise shapes in different representations and different dimensions. The study of shapes is closely connected to the concept of 'grasping space'. This means learning to know, explore and conquer, in order to live, breathe and move with more understanding in the space in which we live (Freudenthal, 1973).

59. To achieve this, we must be able to understand the properties of objects, and the relative positions of objects. We must be aware of how we see things and why we see them as we do. We must learn to navigate through space and through constructions and shapes. This means understanding the relationship between shapes and images or visual representations, such as that between a real city and photographs and maps of the same city. It includes also understanding how three-dimensional objects can be represented in two dimensions, how shadows are formed and must be interpreted, what perspective is and how it functions.

CHANGE AND RELATIONSHIPS

60. Every natural phenomenon is a manifestation of change, and in the world around us a multitude of temporary and permanent relationships among phenomena is observed. Examples are: organisms changing as they grow, the cycle of seasons, the ebb and flow of tides, cycles of unemployment, weather changes and stock exchange indices (eg. the Dow-Jones index). Some of these change processes involve and can be described or modelled by straightforward mathematical functions: linear, exponential, periodic or logistic, either discrete or continuous. But many relationships fall into different categories and data analysis is quite often essential in order to determine the kind of relationship that is present. Mathematical relationships often take the shape of equations or inequalities, but relations of a more general nature (e.g. equivalence, divisibility, inclusion, to mention but a few examples) may appear as well.

61. Functional thinking, that is thinking in terms of and about relationships, is one of the most fundamental disciplinary aims of the teaching of mathematics (MAA, 1923). Relationships may be given a variety of different representations, including symbolic, algebraic, graphical, tabular, and geometrical. Different representations may serve different purposes and have different properties. Hence translation between representations often is of key importance in dealing with situations and tasks.

UNCERTAINTY

62. The present *Information Society* offers an abundance of information. This information is often presented as accurate, scientific and with a degree of certainty. However, in daily life we are confronted with uncertain election results, collapsing bridges, stock market crashes, unreliable weather forecasts, poor predictions for population growth, economic models that don't align, and many other demonstrations of the uncertainty of our world.

63. Uncertainty is intended to suggest two related topics: data and chance, phenomena that are the subject of mathematical study in statistics and probability respectively. Relatively recent recommendations concerning school curricula are unanimous in suggesting that statistics and probability should occupy a much more prominent place than has been the case in the past (MSEB 1990, NCTM 1989, Committee of Inquiry into the Teaching of Mathematics in Schools, 1982, LOGSE 1990, NCTM 2000).

64. Specific mathematical concepts and activities that are important in this area are collecting data, data analysis and data display/visualisation, probability, and inference.

65. We now turn to the most important aspect of the mathematics framework, a discussion of the *competencies* that students bring to bear when attempting to solve problems. These are discussed under the broad heading of *mathematical processes.*

MATHEMATICAL PROCESSES

INTRODUCTION - MATHEMATISATION

66. PISA examines the capacities of students to analyse, reason, and communicate mathematical ideas effectively as they pose, formulate, solve and interpret solutions to mathematical problems in a variety of situations. Such problem solving requires students to use the skills and competencies they have acquired through schooling and life experiences. In PISA, a fundamental process that students use to solve real-life problems is referred to as 'mathematisation'.

[following paragraph and quote as a sidebar]

67. Newton might have been describing mathematisation in his major work, 'Mathematical Principles of Natural Philosophy' when he wrote:

But our purpose is only to trace out the quantity and properties of this force from the phenomena, and to apply what we discover in some simple cases as principles, by which, in a mathematical way, we may estimate the effects thereof in more involved cases (Newton, 1687).

68. The earlier discussion of the theoretical basis for the PISA mathematics framework outlined a five-step description of mathematisation. These steps are shown in Figure 5.

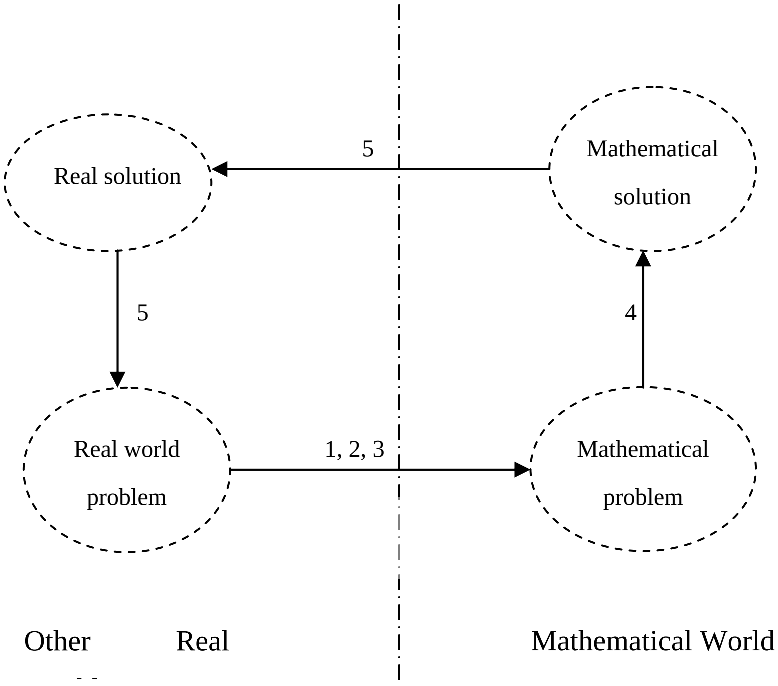


Figure 5 The Mathematisation Cycle

(1) Starting with a problem situated in reality;

(2) Organising it according to mathematical concepts and identifying the relevant mathematics;

(3) Gradually trimming away the reality through processes such as making assumptions, generalising and formalising (which promote the mathematical features of the situation and transform the real-world problem into a mathematical problem that faithfully represents the situation);

(4) Solving the mathematical problem; and

(5) Making sense of the mathematical solution in terms of the real situation, including identifying the limitations of the solution.

69. As the diagram in Figure 5 suggests, the five aspects will be grouped into three clusters.

70. Mathematisation first involves translating the problem from 'reality' into mathematics. This process includes activities such as:

- identifying the relevant mathematics with respect to a problem situated in reality
- representing the problem in a different way; including organising it according to mathematical concepts and making appropriate assumptions
- understanding the relationships between the language of the problem, and symbolic and formal language needed to understand it mathematically
- finding regularities, relations, and patterns
- recognising aspects that are isomorphic with known problems
- translating the problem into mathematics; ie, to a mathematical model

71. As soon as a student has translated the problem into a mathematical form, the whole mathematisation process can continue within mathematics. Students will pose questions like: "Is there…?", "If so, how many?", "How do I find…?", using known mathematical skills and concepts. They will attempt to work on their model of the problem situation, to adjust it, to establish regularities, to identify connections and to create a good mathematical argument. This part of the mathematisation process is generally called the deductive part of the modelling cycle (Schupp, 1988, Blum, 1996). However, other than strictly deductive processes may play a part in this stage. This part of the mathematisation process includes

- using and switching between different representations
- using symbolic, formal and technical language and operations
- refining and adjusting mathematical models; combining and integrating models
- argumentation
- generalisation.

72. The ‘last’ step or steps in solving a problem involve reflecting on the whole mathematisation process and the result. Here students must interpret the results with a critical attitude and validate the whole process. Such reflection takes place at all stages of the process, but it is especially important at the concluding stage of the consideration of the problem. Aspects of this reflecting and validating process are

- understanding the extent and limits of mathematical concepts
- reflecting on mathematical arguments, and explaining and justifying results
- communication of the process and solution
- critiqueing the model and its limits.

73. This stage is indicated in two places in Figure 5 by the label ‘5’, where the mathematisation process passes from the mathematical solution to the real solution, and where this is related back to the original real-world problem.

MATHEMATICAL PROCESSES: THE COMPETENCIES

74. The previous section focused on the major concepts and processes that are involved in mathematisation. An individual who is to engage successfully in mathematisation, within a variety of situations, extra- and intra-mathematical contexts, and overarching ideas, needs to possess a number of mathematical competencies which, taken together, can be seen as constituting comprehensive mathematical competence. Each of these competencies can be possessed at different levels of mastery. Different parts of mathematisation draw differently upon these competencies, both in regard to the particular competencies that are involved and in regard to the level of mastery that needs to be present. To identify and examine these competencies, PISA has decided to make use of eight characteristic mathematical competencies that rely, in their present form, on work of Niss (1999) and his Danish colleagues. Similar formulations may be found in the work of many others (as indicated in Neubrand et al, 2000). It is acknowledged that some of the terms used have different usage among different authors.

1. *Mathematical thinking and reasoning*. This includes posing questions characteristic of mathematics (“Is there...?”, “If so, how many?”, “How do we find...?”); knowing the kinds of answers that mathematics offers to such questions; distinguishing between different kinds of statements (definitions, theorems, conjectures, hypotheses, examples, conditioned assertions); and understanding and handling the extent and limits of given mathematical concepts.

2. *Mathematical argumentation*. This includes knowing what mathematical proofs are and how they differ from other kinds of mathematical reasoning; following and assessing chains of mathematical arguments of different types; possessing a feel for heuristics ("What can(not) happen, and why?"); and creating and expressing mathematical arguments.

3. *Mathematical communication*. This includes expressing oneself, in a variety of ways, on matters with a mathematical content, in oral as well as in written form, and understanding others' written or oral statements about such matters.

4. *Modelling*. This includes structuring the field or situation to be modelled; translating 'reality' into mathematical structures; interpreting mathematical models in terms of 'reality'; working with a mathematical model; validating the model; reflecting, analysing and offering a critique of a model and its results; communicating about the model and its results (including the limitations of such results); and monitoring and controlling the modelling process.

5. *Problem posing and solving*. This includes posing, formulating, and defining different kinds of mathematical problems (for example 'pure', 'applied', 'open-ended' and 'closed'); and solving different kinds of mathematical problems in a variety of ways.

6. *Representation*. This includes decoding and encoding, translating, interpreting and distinguishing between different forms of representation of mathematical objects and situations, and the interrelationships between the various representations; choosing and switching between different forms of representation, according to situation and purpose.

7. *Using symbolic, formal and technical language and operations*. This includes: decoding and interpreting symbolic and formal language, and understanding its relationship to natural language; translating from natural language to symbolic/formal language; handling statements and expressions containing symbols and formulae; using variables, solving equations and undertaking calculations.

8. *Use of aids and tools*. This includes knowing about, and being able to make use of, various aids and tools (including information technology tools) that may assist mathematical activity, and knowing about the limitations of such aids and tools.

75. PISA does not intend to develop test items that assess the above competencies individually. There is considerable overlap among them, and when using mathematics, it is usually necessary to draw simultaneously on many of the competencies, so that any effort to assess individual competencies is likely to result in artificial tasks and unnecessary compartmentalisation of the mathematical literacy domain. The particular competencies

students will be able to display will vary considerably across individuals. This is partially a result of the fact that all learning occurs as a consequence of experiences "with individual knowledge construction occurring through the processes of interaction, negotiation, and collaboration" (De Corte, Greer & Verschaffel, 1996, p. 510). PISA assumes that much of students' mathematics is learned in schools. Understanding of a domain is acquired gradually. More formal and abstract ways of representing and reasoning emerge over time as a consequence of active engagement in activities designed to help informal ideas evolve. Mathematical literacy is also acquired as a consequence of experience involving interactions in a variety of social situations and contexts.

76. In order to be able to productively describe and report students' capabilities, as well as their strengths and weaknesses from an international perspective, some structure is needed. One way of providing this in a comprehensible and manageable way is to describe clusters of competencies, based on the kinds of cognitive demands that are needed to solve different mathematical problems.

MATHEMATICAL PROCESSES: COMPETENCY CLUSTERS

77. PISA has chosen to describe the cognitive activities that these competencies encompass, according to three *competency clusters.*

The 'Reproduction' Cluster

78. The competencies in this cluster essentially involve reproduction of practised knowledge. They include those most commonly used on standardised assessments and classroom tests. These competencies are knowledge of facts and of common problem representations, recognition of equivalents, recalling of familiar mathematical objects and properties, performance of routine procedures, application of standard algorithms and technical skills, manipulation of expressions containing symbols and formulae in standard form, and carrying out computations.

79. Solving simple 'problems' also belongs to this cluster. Such problems include routine 'problems' from textbooks, where the context is mathematical and the translating is straightforward. Items associated with this cluster tend to require mathematical thinking and reasoning that would be limited to recognition and recall of what students have explicitly studied. In terms of mathematical operations, such items typically require only the application of practised routine procedures. They can involve simple mathematical knowledge that most students would be expected to posses, through to more difficult mathematical knowledge and procedures that would be expected of only the more knowledgeable and capable students. The items generally do not require students to justify

their actions or to form mathematical arguments. Only limited demands are made for students to communicate mathematics or shift between different representations of mathematical concepts.

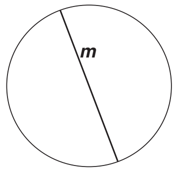
80. Assessment items measuring the Reproduction Cluster competencies might be described with the following key descriptors: reproducing practised material and performing routine operations.

Examples

81. Examples of Reproduction Cluster items follow.

Solve the equation $7x - 3 = 13x + 15$
What is the average of 7, 12, 8, 14, 15, 9?
Write 69% as a fraction

Line ***m*** is called the circle's: ___________



1000 zed is put in a savings account at a Bank, with an interest rate of 4%.
How many zed will there be in the account after one year?

Figure 6 Examples of items illustrating the Reproduction Cluster.

82. In order to make more clear the boundary for items from the Reproduction Cluster, an example is provided that does NOT belong to the Reproduction Cluster:

1000 zed is put in a savings account at a Bank. There are two choices:
one can get a rate of 4% OR one can get a10 zed bonus from the bank
and a 3% rate. Which option is better after one year? After two years?

83. This problem will take most students beyond the simple application of a routine procedure, and requires the application of a chain of reasoning and a sequence of computational steps that are not characteristic of the Reproduction Cluster competencies.

The 'Connections' Cluster

84. The competencies in this cluster include those related to students' planning for problem solving by drawing connections between different mathematical content strands, or between different overarching ideas. They also include students' abilities to combine and integrate information in order to tackle and solve problems. The Connections Cluster of competencies reflects students' abilities to choose and develop strategies, to choose mathematical tools, to use multiple methods or to apply multiple steps in the mathematisation process. These competencies include students' abilities to interpret the meaning of a solution and to check the validity of their work.

85. Problems that reflect student competencies in the Connections Cluster require students to use appropriate elements from different mathematical content areas, or from different overarching ideas, in combination with conceptual thinking and reasoning based on material that does not call for large extensions of where the student has been before.

86. In solving such problems, students may be expected to handle different methods of representation, according to situation and purpose. The mathematical connections required could involve students in being able to distinguish and relate different statements such as definitions, claims, examples, conditional assertions, and proofs.

87. The Connections Cluster competencies build on the Reproduction Cluster competencies in taking problem solving to situations that are not simply routine, but still involve familiar, or quasi-familiar, settings. Solving the problems given frequently requires some reasoning, explanation or argumentation of a mathematical nature. Further, students using Connections Cluster competencies should be able to 'model' the problem, or at least carry out some modelling steps: Students will typically show evidence of their ability to use various representations in solving problems; they are also able to translate the language of the problem into appropriate mathematical language.

88. Items associated with this cluster usually require some evidence of the integration and connection of material from the various overarching ideas, or from different mathematical curriculum strands, or the linking of different representations of a problem. Such items require students to recognise equivalent statements and to operate with a broad range of examples and applications of mathematical concepts and procedures. These items frequently necessitate student construction of and communication of arguments and explanations, and the beginnings of mathematical generalisations. It is necessary for students to translate common situations from the language of the problem situation and context into mathematical symbolism in solving the items posed. The tasks posed typically require students to shift among verbal, graphical, tabular, and symbolic representations of concepts and situations they have experienced in their study of mathematics. The items often require the use of common tools and aids in confronting mathematical situations.

89. Assessment items measuring the Connections Cluster of competencies might be described with the following key descriptors: integrating, connecting, and modest extension of practised material.

Examples

90. Our first example of a Connections Cluster item was given under the Reproduction Cluster. Other examples of Connections Cluster items follow.

Mary lives two kilometres from school, Martin five. How far do Mary and Martin live from each other?

91. When this problem was originally presented to teachers many of them rejected the item for the reason that it was too easy – one could easily see that the answer is 3. Another group of teachers reacted that this was not a good item because there was no answer – they meant there is not one single numerical answer. A third reaction to the item was that it was not a good item because of the fact that there were many possible answers, since without further information the most that can be concluded is that they live somewhere between 3 and 7 kilometres apart, and that is not desirable for an item. A small group thought it was an excellent item, because you have to understand the question, it is real problem solving because there is no strategy known to the student, and it is beautiful mathematics, although you have no clue how they will solve the problem. It is this last interpretation that associates the problem with the Connections Cluster of competencies.

The Office Renting Problem

The following two advertisements appeared in a daily newspaper in a

BUILDING A	**BUILDING B**
Office space available	Office space available
58-95 square metres	35-260 square metres
475 *zeds* per month	90 *zeds* per square

country where the units of currency are *zeds*.

If a company is interested in renting an office of 110 square metres in that country for a year, at which office building, A or B, should the company rent the office in order to get the lower price? Show your work. [© IEA/TIMSS.]

The Pizza Problem

A pizzeria serves two round pizzas of the same thickness in different sizes. The smaller one has a diameter of 30 cm and costs 30 zeds. The larger one has a diameter of 40 cm and costs 40 zeds. [© PRIM, Stockholm Institute of Education.]

Which pizza is better value for money? Show your reasoning.

Figure 7 Examples of items illustrating the Connections Cluster

92. In both of these problems, students are required to translate a real-world situation into mathematical language, to develop a mathematical model that enables a suitable comparison to be made, to check that the solution found fits in with the initial question context and to communicate the result. These are all activities associated with the Connections Cluster.

The 'Reflection' Cluster

93. The competencies in this cluster include an element of reflectiveness on the part of the student about the processes needed or used to solve a problem. They relate to students' abilities to plan solution strategies and implement them in problem settings that contain more elements and may be more 'original' (or unfamiliar) than those in the Connections Cluster. These competencies require students not only to mathematise the problems, but also to develop original solution models. Items measuring the Reflection Cluster of competencies should reflect students' abilities to analyse, to interpret, to reflect, to explain, and to present mathematical generalisations, arguments and proofs. Students should be able to pose problems in addition to solving problems.

94. Items assessing the Reflection Cluster of competencies might sometimes also reflect expectations that students can make connections between mathematics and applications involving problem solving in other disciplines.

95. These competencies need to be complemented by communication skills reflecting students' abilities to both communicate their mathematical ideas as well as to understand the mathematical communication of others.

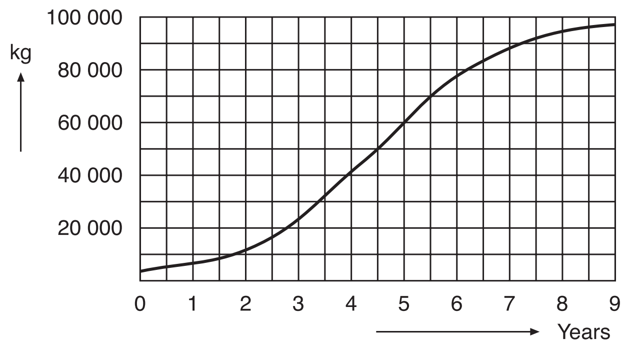
96. Items associated with this cluster require considerable student command of mathematical thinking, reasoning, and argumentation. Such items often require communication of student understanding of and ability to operate with appropriate language, reflecting their abilities to model, solve, and communicate solutions to situations that extend what they have studied in their classroom work with mathematics. Such items involve multi-step problems that make considerable demand for student development of representations and translations as part of the solution process. Performance on such items may reflect not only student knowledge of how to use mathematical tools and aids, but also when to make such use of these tools and aids.

97. Assessment items measuring the Reflection Cluster of competencies might be described with the following key descriptors: advanced reasoning, argumentation, abstraction, generalisation, and modelling applied to new contexts.

Examples

98. Examples of Reflection Cluster items follow.

Some fish were introduced to a waterway. The graph shows a model of the growth in the combined weight of fish in the waterway.



Suppose a fisherman plans to wait a number of years and then start catching fish from the waterway. How many years should the fisherman wait if he or she wishes to maximise the number of fish he or she can catch annually from that year on? Provide an argument to support your answer.

99. It is clear that this problem fits the definition of mathematical problem solving in an authentic context. Students will have to come up with their own strategy and argumentation in a somewhat complex and unfamiliar problem. The complexity lies in part in the need to thoughtfully combine information presented both graphically and in text. But also that there is no answer that students can see immediately – they need to come up with a good and original strategy. They need to interpret the graph, and realise for instance that the rate of growth reaches a maximum after five years or so. To be successful, students need to reflect on their solution as it emerges and think about the success of their strategy. Furthermore the problem asks for an argument and an indication of 'proof'. One possibility is to use the

method of trial and error: see what happens if you wait only 3 years for instance. And go on from there. If you wait till the end of the fifth year then you can have a big harvest every year – 20 000 kg of fish. If you can't wait that long, and start to harvest one year earlier you can catch only 17 000 kg, and if you wait too long (six years) you can only catch 18 000 kg per year. So the optimal result occurs when harvesting commences after five years.

In a certain country, the national defence budget is $30 million for 1980. The total budget for that year is $500 million. The following year the defence budget is $35 million, while the total budget is $605 million. Inflation during the period covered by the two budgets amounted to 10 per cent.

a) You are invited to give a lecture for a pacifist society. You intend to explain that the defence budget decreased over this period. Explain how you would do this.

b) You are invited to lecture to a military academy. You intend to explain that the defence budget increased over this period. Explain how you would do this.

Source: de Lange (1987); see also MSEB (1991). Used with permission.

Figure 8 Examples of items illustrating the Reflection Cluster

100. This problem has been thoroughly researched with 16-year-old students. It illustrates Reflection Cluster problems very well: the students recognised the literacy aspect immediately and quite often were able to do some kind of generalisation as the heart of the solution lies in the recognition that the key mathematical concepts here are absolute and relative growth. The inflation can of course be left out to make the problem more accessible for somewhat younger students without losing the key conceptual ideas behind the problem. But one loses in the complexity and thus in the required mathematisation. Another way to make the item 'easier' is to present the data in a table or schema. These mathematisation aspects are then not any more actions students have to carry out – they can start right away at the heart of the matter.

SUMMARY OF MATHEMATICAL PROCESSES IN PISA MATHEMATICS

101. Figure 9 provides a diagrammatic representation of the competency clusters and summarises the distinctions between them.

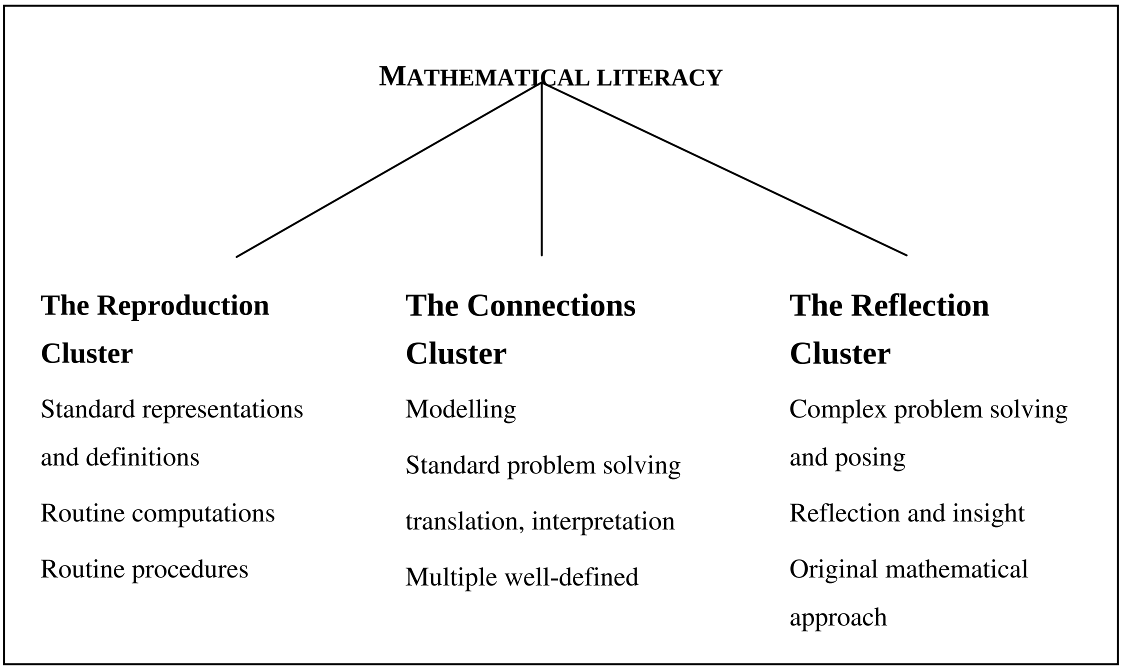


Figure 9: Diagrammatic Representation of the Competency Clusters

102. The easiest assessment items to classify are those involving Reproduction Cluster competencies. They include simple problems that only require recall. Reproduction and routine skills fall into this class. While some context may be involved in a problem, the solution path is so straightforward that it is still classified as a routine process, requiring almost no mathematisation. Key descriptors for this cluster are standard representations and definitions; routine computations, procedures and problem solving.

103. Figure 9 shows that problems involving Connections Cluster competencies require some reasoning, interpretation and reflection and making connections and/or integration. Key descriptors for this class: Standard problem solving (making and using connections, integration) involving: reasoning, reflection, interpretation.

104. Items involving Reflection Cluster competencies require mathematical insight, have unusual contexts or original methods and may involve generalisation as the highest level of mathematisation. Strategy development by students is needed for assessment items that involve competencies in both Connections and Reflection competency clusters, the difference again is the level of complexity or originality. Key descriptors for this cluster: Original and complex problem solving involving insight, interpretation, reflection and possibly generalisation.

ASSESSING MATHEMATICAL LITERACY

TASK CHARACTERISTICS

105. In the previous sections, the PISA mathematical literacy domain has been defined and the structure of the assessment framework has been described. This section considers in more detail features of the assessment tasks that will be used to assess students. The nature of the tasks and the task formats are described.

The nature of tasks for PISA Mathematics

106. PISA is an international test of the literacy skills of 15-year-olds. All test items used should be suitable for the population of 15-year-old students in OECD countries.

107. In general, items will consist of some stimulus material or information, an introduction, and the actual question. In addition, for non-multiple choice items, a detailed coding scheme will be developed to enable trained markers across the range of participating countries to code the student responses in a consistent and reliable way.

108. In an earlier section of this framework, the situations to be used for PISA Mathematics items were discussed in some detail. For PISA 2003, each item will be set in one of five situation-types: personal, educational, occupational, public, and scientific. The items selected for the PISA 2003 mathematics test instruments will represent a spread across these situation types.

109. In addition, item contexts that can be regarded as *authentic* will be preferred. That is, PISA values most highly tasks that could be encountered in one of a variety of real-world situations, and that have a context for which the use of mathematics to solve the problem would be authentic. Problems with extra-mathematical contexts that influence the solution and its interpretation are preferred as a vehicle for assessing mathematical literacy.

110. Items should relate predominantly to one of the phenomenological problem categories (the overarching ideas) described in the framework. The selection of mathematics test items for PISA 2003 will ensure that the four overarching ideas are represented about equally.

111. Items should embody one or more of the mathematical processes that are described in the framework, and should be identified predominantly with one of the competency clusters. Items reflecting the three competency clusters (Reproduction, Connections, Reflection) will be selected for inclusion in the PISA 2003 test instrument in the ratio of about 1:2:1 respectively.

112. The level of reading required to successfully engage with an item will be considered very carefully in the development and selection of items for inclusion in the PISA 2003 test instrument. The wording of items will be as simple and direct as possible. Care will also be taken to avoid question contexts that would create a cultural bias.

113. Items selected for inclusion in the PISA test instruments will represent a broad range of difficulties, to match the expected wide ability range of students participating in the PISA assessment from across OECD countries. In addition, the major classifications of the framework (particularly competency clusters, and overarching ideas) should as far as possible be represented with items of a wide range of difficulties. Item difficulties will be established in an extensive Field Trial of test items prior to item selection for the main PISA survey.

Task formats

114. When assessment instruments are devised, the impact of the format of the tasks on student performance, and hence on the definition of the construct that is being assessed, must be carefully considered. This issue is particularly pertinent in a project such as PISA in which the large-scale cross-national context for testing places serious constraints on the range of feasible item formats.

115. PISA will assess mathematical literacy through a combination of items with open-constructed response formats, closed-constructed response formats and multiple-choice formats. About equal numbers of each of these item-formats will be used in constructing the test instruments for PISA 2003.

116. Based on experience in developing and using test items for PISA 2000, the use of multiple-choice formats is generally regarded as most suitable for assessing items that would be associated with the Reproduction and Connections competency clusters. For an example of this item-type, see Figure 10 which shows an item that would be associated with the Connections competency cluster. To solve this problem, students must translate the problem into mathematical terms, devise a model to represent the periodic nature of the context described, and extend to the pattern to match the result with one of the given options.

A seal has to breathe even if it is asleep. Martin observed a seal for one hour. At the start of his observation the seal dove to the bottom of the sea and started to sleep. In 8 minutes it slowly floated to the surface and took a breath.

In 3 minutes it was back at the bottom of the sea again and the whole process started over in a very regular way.

After one hour the seal was:

a) at the bottom
b) on its way up
c) breathing
d) on its way down

Figure 10 Example item with a limited number of defined response-options

117. For any higher-order goals and more complex processes, other test formats will often be preferred. Closed constructed-response items pose similar questions as multiple-choice items, but students are asked to produce a response that can be easily judged to be either correct or incorrect. For items in this format, guessing is not likely to be a concern, and the provision of distractors (which influence the construct that is being assessed) is not necessary. For example, for the problem in Figure 11 there is one correct answer and many possible incorrect answers.

> Tepla Loroupe won the 1998 Rotterdam marathon. "It was easy", she said, "the course was quite flat".
>
> Here you see a graph of the differences in elevation of the Rotterdam marathon course:
>
> [***Differences in level of the course — in metres relative to the starting point***]
>
> What was the difference between the highest and the lowest points of the course?

Figure 11 Example item with one correct answer and many incorrect answers

118. Open constructed-response items require a more extended response from the student, and the process of producing a response frequently involves higher-order activities. Often such items not only ask the student to produce a response, but also require the student to show the steps taken or to explain how the answer was reached. The key feature of open constructed-response items is that they allow students to demonstrate their abilities by providing solutions at a range of levels of mathematical complexity. The item in Figure 12 is an example.

Indonesia lies between Malaysia and Australia. Some data of the population of Indonesia and its distribution over the islands is shown in the following table:

Region	Surface area (km2)	Percentage of total area	Population in 1980 (millions)	Percentage of total population
Java/Madura	132 187	6.95	91 281	61.87
Sumatra	473 606	24.86	27 981	18.99
Kalimantan (Borneo)	539 460	28.32	6 721	4.56
Sulawesi (Celebes)	189 216	9.93	10 377	7.04
Bali	5 561	0.30	2 470	1.68
Irian Jaya	421 981	22.16	1 145	5.02
TOTAL	1 905 569	100.00	147 384	100.00

One of the main challenges for Indonesia is the uneven distribution of the population over the islands. From the table we can see that Java, which has less than 7% of the total area, has almost 62% of the population.

Question: Design a graph (or graphs) that shows the uneven distribution of the Indonesian population.

Source: de Lange et Verhage (1992). Used with permission.

Figure 12 Example item that requires a constructed response

119. For PISA, about one third of the mathematics items will be open constructed-response items. The responses to these items require coding by trained people who implement a coding rubric that may require an element of professional judgement. Because of the potential for disagreement between markers of these items, PISA will implement marker reliability studies to monitor the extent of disagreement. Experience in these types of studies shows that clear coding rubrics can be developed and reliable scores can be obtained.

120. PISA will make some use of a task format in which several items are linked to common stimulus material. Tasks of this format give students the opportunity to become involved with a context or problem by asking a series of questions of increasing complexity. The first few questions are typically multiple-choice or closed-constructed items while subsequent items are typically open-constructed items. This format can be used to assess each of the competency clusters.

121. One reason for the use of common stimulus task formats is that it allows realistic tasks to be devised and the complexity of real-life situations to be reflected in them. Another reason relates to the efficient use of testing time, cutting down on the time required for a student to ‘get into’ the subject matter of the situation. The necessity of making each scored point independent of others within the task is recognised and taken into account in the

design of the PISA tasks and of the response coding and scoring rubrics. The importance of minimising bias that may result from the use of fewer situations is also recognised.

ASSESSMENT STRUCTURE

122. The PISA 2003 test instruments will contain a total of 240 minutes of testing time, which will mean provision of up to 130 mathematics items. It is expected that these will be arranged in seven clusters of items, with each item-cluster representing 30 minutes of testing time. The item-clusters will be placed in test booklets according to a rotated test design.

123. The total testing time for mathematics will be distributed as evenly as possible across the four overarching ideas (Quantity, Space and Shape, Change and Relationships, and Uncertainty), and the five situations described in the framework (personal, educational, occupational, public, and scientific). The proportion of items reflecting the three competency clusters (Reproduction, Connections and Reflection) will be about 1:2:1. About one-third of the items will be in multiple-choice format, about one-third in closed constructed response format, and about one-third in open constructed response format.

124. A recommended division of 120 items that would meet these expectations, broadly similar to the division used for the PISA 2000 test instrument, is summarised in Table 1 following.

ITEM TYPE	COMPETENCY CLUSTER			
	REPRODUCTION	CONNECTIONS	REFLECTION	**TOTAL**
MULTIPLE CHOICE	About 15	About 20	About 5	**About 40**
CLOSED CONSTRUCTED RESPONSE	About 10	About 25	About 5	**About 40**
OPEN CONSTRUCTED RESPONSE	About 5	About 15	About 20	**About 40**
TOTAL	**About 30**	**About 60**	**About 30**	

Table 1 Expected division of Mathematics items for PISA 2003

REPORTING MATHEMATICAL PROFICIENCY

125. To summarise data from responses to the PISA test instruments, a five level described performance scale will be created (Masters & Forster, 1996; Masters, Adams, & Wilson, 1999). The scale will be created statistically using an Item Response Modelling approach to scaling ordered outcome data. The overall scale will be used to describe the

nature of performance by classifying the nations in terms of the five levels of overall performance, and thus provide a frame of reference for international comparisons.

126. Consideration will be given to developing a number of separate reporting scales. Such sub-scales could be based on the three competency clusters, or on the four overarching ideas. Decisions about the development of separate reporting scales will be made on a variety of grounds, including psychometric considerations, following analysis of the data generated by the PISA assessments. To facilitate these possibilities, it will be necessary to ensure that sufficient items are selected for inclusion in the PISA test instrument from each potential reporting category. Moreover, items within each such category will need to have a suitably wide range of difficulties.

127. The competency clusters described earlier in this framework reflect conceptual categories of increasing complexity, but do not strictly reflect a hierarchy of student performances based on item difficulty. Conceptual complexity is only one component of difficulty that can be used to determine levels of performance. Others include familiarity, recent opportunity to learn, practice, and so forth. Thus, a multiple-choice item involving competencies from the Reflection Cluster (e.g. 'which of the following is a rectangular parallelepiped?' followed by pictures of a ball, a can, a box, and a square) may be very difficult for many students because of lack of familiarity with the term. If students had similar background and experience with the contexts and content of a set of test items, competency clusters and a derived performance scale would be very highly correlated. However this is rarely the case. Nevertheless, one would still expect a positive relationship between competency clusters and overall performance levels, but some tasks designed to reflect the same competency cluster will be associated with different performance levels.

128. Factors that will underpin increasing levels of mathematical proficiency include those summarised as follows:

- The kind and degree of interpretation and reflection required. This includes the nature of demands arising from the problem context; the extent to which the mathematical demands of the problem are apparent or to which students must impose their own mathematical construction on the problem; and the extent to which insight, complex reasoning and generalisation are required.

- The kind of representation skills that are necessary, ranging from problems where only one mode of representation is used, to problems where students have to switch between different modes of representation or to find appropriate modes of representation themselves.

- The kind and level of mathematical skill required, ranging from single-step problems requiring students to reproduce basic mathematical facts and simple computation processes through to multi-step problems involving more advanced mathematical knowledge, complex decision-making, information processing, and problem solving and modelling skills.

- The kind and degree of mathematical argumentation that is required, ranging from problems where no arguing is necessary at all, through problems where students may apply well-known arguments, to problems where students have to create mathematical arguments or to understand other people's argumentation or judge the correctness of given arguments or proofs.

129. At the lowest described proficiency level, students typically carry out single-step processes that involve recognition of familiar contexts and mathematically well-formulated problems, reproducing well-known mathematical facts or processes, and applying simple computational skills.

130. At higher proficiency levels, students typically carry out more complex tasks involving more than a single processing step, and they combine different pieces of information or interpret different representations of mathematical concepts or information, recognising which elements are relevant and important. They typically work with given mathematical models or formulations, which are frequently in algebraic form, to identify solutions, or they carry out a small sequence of processing or calculation steps to produce a solution.

131. At the highest proficiency level, students take a more creative and active role in their approach to mathematical problems. They typically interpret more complex information and negotiate a number of processing steps. They typically produce a formulation of a problem and often develop a suitable model that facilitates solution of the problem. Students at this level typically identify and apply relevant tools and knowledge frequently in an unfamiliar problem context, they typically demonstrate insight to identify a suitable solution strategy, and display other higher order cognitive processes such as generalisation, reasoning and argumentation to explain or communicate results.

AIDS AND TOOLS

132. The PISA policy with regard to the use of calculators and other tools is that students should be free to use their own calculators and other tools as they are normally used in school.

133. This represents the most authentic assessment of what students can achieve, and will provide the most informative comparison of the performance of education systems. A

system's choice to allow students to access and use calculators is no different, in principle, from other instructional policy decisions that are made by systems and are not controlled by PISA.

134. Students who are used to having a calculator available to assist them in answering questions if they chose to use such a tool will be disadvantaged if this resource is taken away.

CONCLUSION

135. The aim of the OECD/PISA study is to develop indicators that show how effectively countries have prepared their 15-year-olds to become active, reflective and intelligent citizens from the perspective of their uses of mathematics. To achieve this, OECD/PISA has developed assessments that focus on determining the extent to which students can use what they have learned.

136. This framework provides a definition of mathematical literacy, and sets the context for the assessment of mathematical literacy in 2003 that will permit OECD countries to monitor some important outcomes of their education systems. The definition of mathematical literacy chosen for this framework is consistent with those definitions for literacy in reading and science literacy, and with the PISA orientation of assessing students' capacities to become active and contributing members of society.

137. The major components of the mathematics framework, consistent with the other OECD/PISA frameworks, include contexts for the use of mathematics, mathematical content and mathematical processes, each of which flows directly out of the literacy definition. The discussions of context and content emphasise features of the problems that confront students as citizens, while the discussions of processes emphasise the competencies that students bring to bear to solve those problems. These competencies have been grouped into three so called 'competency clusters' to facilitate a rational treatment of the way a very complex set of cognitive processes are played out within a structured assessment program.

138. The emphasis of the PISA mathematics assessments on making use of one's mathematical knowledge and understanding to solve problems that arise out of one's day-to-day experience embodies an ideal that is achieved to varying degrees in different education systems around the world. The PISA assessments attempt to provide a variety of mathematical problems, with varying degrees of built-in guidance and structure, but pushing towards authentic problems where students must do the thinking themselves.